Journal of Sound and Vibration (2002) **251**(4), 761–765
doi:10.1006/jsvi.2001.3865, available online at http://www.idealibrary.com on IDEAL®

POLAR AXISYMMETRIC VIBRATION OF A HOLLOW TOROID USING THE DIFFERENTIAL QUADRATURE METHOD

W. JIANG AND D. REDEKOP

Department of Mechanical Engineering, University of Ottawa, 770 *King Edward Avenue, P.O. Box* 450, *Station A, Ottawa, Canada K*1*N* 6*N*5

(*Received* 6 *July* 2001)

1. INTRODUCTION

In recent years, work on toroidal shell vibrations has been based on shell theory [1–3]. A comment by Gall *et al.* [4] that some vehicle tires have a thickness placing them at the limit of shell theory has rekindled interest in the use of the theory of elasticity [5] for the analysis of these shells.

In this work, the theory of elasticity is used to study the polar axisymmetric vibrations of isotropic toroidal shells with circular cross-section and uniform thickness. The theory is valid for shells of any thickness and can serve to assess the accuracy of the shell theories. Governing equations developed earlier by McGill and Lenzen [6] are adapted for the analysis. Numerical results are found using the new differential quadrature method (DQM). The solution is partially validated by comparing with results from the finite element method (FEM). The current work represents the initial stage of a program intended to cover dynamics of non-symmetric anisotropic toroidal shells of variable thickness.

2. AXISYMMETRIC TOROIDAL ELASTICITY THEORY

The hollow toroid has a bend radius R, an annular cross-section of mean radius $r_0 = (r_{\mathrm{I}} + r_{\mathrm{II}})/2$, and a thickness $h = r_{\mathrm{I}} - r_{\mathrm{II}}$. The shell is axisymmetric and complete, and thus extends through 360° in the circumferential (ϕ) and meridional (θ) directions.

The equations of motion for polar axisymmetric vibrations, adapted from the theory of McGill and Lenzen [6], can be written as

$$L_{11}u_r + L_{12}u_\theta + \beta^2 u_r = 0, \quad L_{21}u_r + L_{22}u_\theta + \beta^2 u_\theta = 0, \tag{1}$$

where

$$L_{11} = f_1\frac{\partial^2}{\partial r^2} + f_2\frac{\partial}{\partial r} + f_3\frac{\partial^2}{\partial \theta^2} + f_4\frac{\partial}{\partial \theta} + f_5, \quad L_{12} = f_6\frac{\partial}{\partial r} + f_7\frac{\partial}{\partial \theta} + f_8\frac{\partial^2}{\partial r\partial \theta} + f_9,$$

$$L_{21} = f_{10}\frac{\partial}{\partial \theta^2} + f_{11}\frac{\partial^2}{\partial r\partial \theta} + f_{12}, \quad L_{22} = f_{13}\frac{\partial^2}{\partial r^2} + f_{14}\frac{\partial}{\partial r} + f_{15}\frac{\partial^2}{\partial \theta^2} + f_{16}\frac{\partial}{\partial \theta} + f_{17}.$$

The displacement components u_r, u_θ are in the r, θ directions, respectively, the f_i are known functions of r, θ given in reference [7], $\beta^2 = \hat{\rho} r_0^2\omega^2/(\lambda + 2\mu)$, $\hat{\rho}$ is the mass density, ω is the natural frequency in Hz, $\lambda = 2\mu v/(1 - 2v)$, $2\mu = E/(1 + v)$, E is the Young's modulus and

0022-460X/02/140761 + 05 $35.00/0

ν is the Poisson ratio. The stresses are given in terms of the displacements by

$$\sigma_r = \lambda\left[g_1\frac{\partial u_r}{\partial r} + g_2 u_r + g_3\frac{\partial u_\theta}{\partial \theta} + g_4 u_\theta\right], \quad \sigma_\theta = \lambda\left[g_5\frac{\partial u_r}{\partial r} + g_6 u_r + g_7\frac{\partial u_\theta}{\partial \theta} + g_8 u_\theta\right],$$

$$\sigma_{r\theta} = \mu\left[g_9\frac{\partial u_r}{\partial \theta} + g_{10}\frac{\partial u_\theta}{\partial r} + g_{11} u_\theta\right], \tag{2}$$

where the g_i are known functions given in reference [7]. The solution to the governing equations is subject to the boundary conditions $\sigma_r = \sigma_{r\theta} = 0$ on the surfaces $r = r_{\mathrm{I}}$, $r = r_{\mathrm{II}}$.

3. DIFFERENTIAL QUADRATURE METHOD

The DQM approach is used to determine the numerical results. This method which has developed a reputation for high precision was introduced to vibration problems of thin shells by Bert and Malik [8]. In the current study, the DQM is adapted to cover the requirements of the theory of elasticity. A meshing is required in the cross-section defined by the co-ordinates r and θ. An extended exposition of the DQM approach is given by Bert and Malik [8] and thus only a brief outline is given in the following.

The basis of the DQM is the meshing of the domain and the representation in the domain of the derivatives of a function $f(x)$ by a weighted sum of trial function values, i.e.,

$$\left.\frac{\mathrm{d}^k f}{\mathrm{d}x^k}\right|_{x=x_i} = \sum_{j=1}^{M} A_{ij}^{(k)} f(x_j), \tag{3}$$

where the $A_{ij}^{(k)}$ are the weighting coefficients of the kth order derivative at the ith sampling point of the mesh in the x direction, and M is the number of sampling points in this direction. Values are determined *a priori* for the weighting coefficients depending on the choice of the trial functions. In the analysis, either the DQM analogue of a governing equation for the domain or a boundary equation is represented at each sampling point.

For the current problem polynomial trial functions [8] are used for the radial direction as

$$f(r) = 1, r, r^2, \ldots, r^{M-1}. \tag{4}$$

The Chebyshev–Gauss–Lobatto spacing [8] is used for the sampling points in this direction of the mesh, and explicit formulas are then available for the weighting coefficients. Harmonic trial functions [8] are used in the meridional direction so as to satisfy continuity conditions across $\theta = 180, 360°$. The trial functions are thus taken as

$$f(\theta) = \cos[2(k-1)\pi\phi], \quad k = 1, 2, 3, \ldots, N/2+1,$$

$$f(\theta) = \sin[2(k-N/2-1)\pi\phi], \quad k = N/2+2, N/2+3, \ldots, N, \tag{5}$$

where N is an even number. Again formulas are available for the weighting functions involved.

Use of the quadrature rules (3)–(5) for the derivatives in the governing equations (1) and the boundary conditions leads to the transformed DQM domain and boundary equations. The assembly of these equations yields a matrix equation of the form

$$\begin{bmatrix} S_{bb} & S_{bd} \\ S_{db} & S_{dd} \end{bmatrix}\begin{Bmatrix} \Delta_b \\ \Delta_d \end{Bmatrix} - \beta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \Delta_b \\ \Delta_d \end{Bmatrix} = \{0\}, \tag{6}$$

TABLE 1

Convergence of FEM and DQM results

Mode	1	2	3	4	5
FEM					
6×24	0·28599	0·50139	1·18873	1·48968	2·73690
12×24	0·06991	0·20511	0·88066	0·91887	1·43433
12×48	0·06991	0·20510	0·88060	0·91878	1·43431
24×48	0·06991	0·20509	0·88060	0·91877	1·43431
DQM					
6×24	0·07025	0·21195	0·88464	0·93741	1·43588
12×24	0·06989	0·20487	0·88016	0·91812	1·43477
12×48	0·06989	0·20487	0·88016	0·91812	1·43477
24×48	0·06989	0·20487	0·88016	0·91812	1·43477

where vectors Δ_b and Δ_d contain the displacements u_r, u_θ corresponding to the boundary and domain sampling points respectively. The vector Δ_b is eliminated using the static condensation technique. The matrix equation then reduces to

$$([-S_{db}S_{bb}^{-1}S_{bd} + S_{dd}] - \beta^2)\{\Delta_d\} = \{0\}. \tag{7}$$

This equation represents a standard eigenvalue problem and can be used to find the frequency parameters β^2. The theory presented in the preceding was coded in the $MATLAB^{TM}$ program axifre.m. Results from this program are given in the following.

4. VALIDATION AND RESULTS

Two problems were considered to obtain partial validation for the analysis procedure. In the first problem, DQM results are compared for a thick toroid with results from the FEM code ADINA [9]. An axisymmetric rather than a shell theory element was used. The shell parameters were: $r_{\mathrm{I}}/r_0 = 1{\cdot}3333$, $r_{\mathrm{II}}/r_0 = 0{\cdot}6667$, $R = 5{\cdot}0$, $\nu = 0{\cdot}3$. Table 1 gives a comparison of the frequency parameter β^2 for the first five natural frequencies for a number of different meshes. It is seen that there is rapid convergence for both methods and close agreement for the finer meshes. The DQM as a specialized solution has a major computer time advantage over the FEM.

In the second validation problem, DQM results are compared for a thin toroid with results obtained from the FEM. Table 2 gives a comparison of the results obtained for the frequency parameter $\alpha = \hat{\rho}\omega^2R^2/E$. The lowest axisymmetric frequencies for three cases cited in reference [1] are presented. The thin shell theory results of reference [1] are also given. It is seen that there is close agreement between the three methods, especially for the very thin shells.

Additional results were computed to indicate the variation of the natural frequencies with the Poisson ratio. The geometric parameters were $r_{\mathrm{I}}/r_0 = 1{\cdot}01$, $r_{\mathrm{II}}/r_0 = 0{\cdot}99$, $R = 2{\cdot}0$. Table 3 indicates the variation of the frequency parameter α with ν as computed by the FEM and DQM methods. The values given represent the first five antisymmetric modes for a toroid described in reference [6]. Results from the two methods agree very closely with each other, but are at major variance with the results given in Figure 7 of reference [6].

TABLE 2

Comparison with shell theory and FEM results

Case	Shell theory [1]	FEM Mesh	FEM Value	DQM Mesh	DQM Value
$R/r_0 = 10$		2×400	0·0149	6×18	0·0149
$h/r_0 = 0{\cdot}01$	0·0148	2×800	0·0149	8×24	0·0149
		4×800	0·0149	12×36	0·0148
$R/r_0 = 10$		2×400	0·0322	6×18	0·0319
$h/r_0 = 0{\cdot}02$	0·0315	2×800	0·0322	8×24	0·0319
		4×800	0·0322	12×36	0·0318
$R/r_0 = 5$		2×500	0·0153	6×18	0·0152
$h/r_0 = 0{\cdot}02$	0·0150	2×1000	0·0153	8×24	0·0152
		4×1000	0·0153	12×36	0·0152

TABLE 3

Variation of frequency parameter with ν

Mode	$\nu = 0{\cdot}25$ FEM	$\nu = 0{\cdot}25$ DQM	$\nu = 0{\cdot}30$ FEM	$\nu = 0{\cdot}30$ DQM	$\nu = 0{\cdot}35$ FEM	$\nu = 0{\cdot}35$ DQM
1	0·001287	0·001287	0·001165	0·001165	0·000996	0·000996
2	0·052849	0·052849	0·047238	0·047238	0·039773	0·039773
3	0·065252	0·065251	0·058588	0·058587	0·049589	0·049588
2	0·097698	0·097697	0·087818	0·087817	0·074455	0·074454
3	0·138559	0·138556	0·125141	0·125138	0·106669	0·106666

5. CONCLUSION

Accurate solutions have been presented for the natural polar axisymmetric frequencies of isotropic toroidal shells of arbitrary uniform thickness. These values are useful in validating shell theory and finite element procedures. Work is currently underway to extend the current method to the non-symmetric dynamics problem of anisotropic toroidal shells of variable thickness.

REFERENCES

1. T. BALDERES and A. E. ARMENAKAS 1973 *American Institute of Aeronautics and Astronautics Journal* **11**, 1637–1644. Free vibrations of ring-stiffened toroidal shells.
2. F. KOSAWADA, K. SUZUKI and S. TAKAHASHI 1986 *Bulletin JSME* **29**, 3036–3042. Free vibrations of thick toroidal shells.
3. A. Y. T. LEUNG and N. T. C. KWOK 1994 *Thin-Walled Structures* **18**, 317–332. Free vibration analysis of a toroidal shell.
4. R. GALL, F. TABADDOR, D. ROBBINS, P. MAJORS, W. SHEPERD and S. JOHNSON 1995 *Tire Science Technology* **23**, 175–188. Some notes on the finite element analysis of tires.

5. D. REDEKOP 1992 *International Journal of Pressure Vessels and Piping* **51**, 189–209. A displacement solution in toroidal elasticity.
6. D. J. MCGILL and K. H. LENZEN 1967 *SIAM Journal* **15**, 678–692. Polar axisymmetric free oscillations of thick hollowed tori.
7. W. JIANG *M. A. Sc. Thesis*, *University of Ottawa*, in progress. Three-dimensional analysis of anisotropic toroids using the DQM.
8. C. W. BERT and M. MALIK 1996 *American Society of Mechanical Engineering Journal of Pressure Vessel Technology* **118**, 1–12. Free vibration analysis of thin cylindrical shells by the differential quadrature method.
9. 1997 *ADINA Verification Manual*, *ARD* 97-9. Watertown, MA: ADINA R&D Inc.